Operation Research and Game Theory: Part 2 – Game Theory.

FCS, 2024. Homework 1.

- The solution should be sent till 23.59, May 5th, 2024, to the e-mail **games.hse.sandomir@gmail.com**.
- The solution should be in the unique Pdf file (scan, latex, word, good-quality photo converted to pdf). Please, use the name ORGT_HW1_your_surname.pdf.
- The handwriting should be legible.
- The discount for the late assignment is 33% per day.

1. (15 points)

   Find all Nash equilibria (in pure and mixed strategies) in the following game

	a	b	c
t	0, 4	5, 6	8, 8
b	3, 9	6, 5	5, 1

2. (15 points)

   The presidential elections are coming in the country N. There are two candidates: strong and weak. The candidate's strategy is her election program: left `L`, centrist `C`, or right `R`. The payoff matrix is as follows:

		Weak		
		L	C	R
Strong	L	$1, 0$	α,$1-\alpha$	$1-\alpha$,α
	C	$1-\alpha$,α	$1, 0$	$1-\alpha$,α
	R	$1-\alpha$,α	α,$1-\alpha$	$1, 0$

   Let $0 < \alpha \leq \frac{1}{2}$. If a weak candidate chooses the same strategy as a strong candidate, she will lose outright; if she chooses another, she will lose by a smaller margin (which is better), or may even win. Find Nash equilibrium. Explain how it varies with α.

3. (20 points)

   The population of the country N is engaged in one of two types of activities: work and produce goods, or produce the offspring and live on welfare, which is paid from the taxes collected from workers. Let $V \in [0, 1]$ be the share of workers. Let α be the maximal profit of a worker, $\beta < \alpha$ be her guaranteed gain (it is not taxable). Thus, the maximal total sum of taxes is $(\alpha - \beta)V$.

   Let m be the maximal amount of a subsidy for one person. But the exact value depend on the size of tax fund and the number of "parents". Therefore, the maximal size of the tax fund that is needed is $m(1 - V)$. If $(\alpha - \beta)V > m(1 - V)$ the the size of a subsidy is exactly m and not all available money of workers will be collected by taxes. If $(\alpha - \beta)V < m(1 - V)$, the subsidy will be lower than m (calculate) and a worker will obtain exactly β.

(a) How do the gains of a worker and a "parent" depend on V?

(b) Consider the game, in which every person decides whether to work or to produce offspring (assume that the individual decision doesn't change the share V). Find Nash equilibrium in this game. How does the number of equilibrium depend on the parameters of the model? Explain why different equilibria are possible.

4. (15 points)

Consider the market where 10 firms compete in quantities. At the first stage, 4 firms simultaneously and independently choose their production levels $q_i, i = 1, \ldots, 4$. At the second stage, the rest 6 firms observe the previous quantities and simultaneously and independently choose their own q_j, $j = 5, \ldots, 10$. After all q_i's have been chosen, the market price for the homogeneous product is formed $p(\mathbf{q}) = 100 - \sum_{i=1}^{10} q_i$. Assume also that the per-item production costs for 1-4th firms are the same $0 < c_1 < 100$, and for 5-10th firms are the same $0 < c_2 < 100$. You may assume that all costs are low enough (not to deal with boundary cases). So, the utility of every firm is

$$u_i(\mathbf{q}) = q_i p(\mathbf{q}) - c_i q_i.$$

Find the subgame perfect equilibrium. Calculate the equilibrium price.

5. (15 points)

In some country, a president has just been elected. The actions of the president at the post are determined by the value $r \in [0, 1]$. The gain of the president is r, the gain of the public $1 - r$. For example, r can reflect the share of his personal time that the president spends on non-state affairs, or the amount of money he steals from the state budget (the more he steals, the more r is). Immediately after the election, the public sets the following condition for the president: he will be re-elected only if $r \leq \bar{r}$.

(a) Consider the following two-period game. First, the public sets $\bar{r}$. Then the president chooses r. If $r > \bar{r}$, then the president loses the next election and his gain is r. If $r \leq \bar{r}$, then the president wins the next election and his total gain is $r + \delta R$, where $R > 0$ is the gain from re-election, $\delta > 0$ is the discount (here both are given constants). Find the subgame perfect equilibrium in this game.

(b) Now consider the infinite game: if the president is re-elected then the public can again set $\bar{r}$. After this the president chooses r. If the president is not re-elected then the game stops.

Let the president's gain is $U_p = \sum_{t=0}^{\infty} r_t \delta^t$, where r_t is an action of president at moment t. The public's gain is $1 - r_t$ at every moment t.

Find a stationary equilibrium, i.e. the equilibrium in which $\bar{r}$ and r are the same in all periods.

6. (20 points)

Consider a model with an infinite number of periods $t = 1, 2, \ldots$, in which two firms compete in prices. Demand on the market is subject to cyclical fluctuations: in periods with odd numbers, demand is $d(p) = 1 - p$, in periods with even numbers $d(p) = k - p$, where $k > 1$. For simplicity, we assume that firms have zero costs; firms also have a common discount factor δ. Find under which conditions on the parameters firms will be able to maintain the cooperative equilibrium (monopoly price) using a trigger strategy in which deviation from the monopoly price entails a transition to the minimax strategy until the end of the game. Compare with the case where market demand is constantly low ($k = 1$); explain the difference in results.